

Sobolev inequality for localization of pseudo-relativistic energy

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Abstract. In this article we present Sobolev-type inequalities for the localization of pseudo-relativistic energy.

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In this article we present Sobolev-type inequalities for the localization of pseudo-relativistic energy. This is a continuation of our previous article ([BT]), where a Kato-type inequality for the same localization has been established. In a seminal paper by Lieb and Yau ([LY]) some inequalities for pseudo-relativistic energy localization have been used to establish the stability of matter. We hope that our results can also be used in this area. Another reason for studying such quadratic forms comes from scale-space theory in image processing. It has been shown ([FS]) that pseudo-relativistic energy generates better scale-space theory for \mathbb{R}^2 . Since real images are defined on bounded domains we need similar scale-space theory for such domains. There are two possibilities to do this. One is to take the square root of the Laplacian ([DFFP]). The other possibility is to consider the quadratic form for an image $f(x)$ defined by

$$\int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^2}{|x - y|^3} dx dy.$$

The second possibility is preferable since it is easy to calculate and explicitly involves image contrast. Applications of results obtained in this paper for image processing will be studied in a future article.

Our main result is

Theorem 1. *Let Ω be a bounded domain in \mathbb{R}^n such that $\Omega = \phi(B_1)$, where $B_1 \subset \mathbb{R}^n$, $n \geq 2$ is a unit ball and $\phi : B_1 \rightarrow \Omega$ is a diffeomorphism with*

$$c_1^{-1}|x - y| \leq |\phi(x) - \phi(y)| \leq c_1|x - y| \quad (1)$$

for some constant $c_1 = c_1(\Omega) > 1$. Then there exists a constant $c_2 = c_2(\Omega) > 0$ such that

$$\int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{n+1}} dx dy \geq c_2 \left(\int_{\Omega} |f(x)|^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}}$$

for any continuous function $f : \Omega \rightarrow \mathbb{C}$ with $\text{supp } f \subset \Omega$.

The proof of Theorem 1 will be preceded by the proofs of two auxiliary lemmas.

Lemma 1. *Let $\Omega \subset \mathbb{R}^n$ either satisfies the conditions of Theorem 1 or $\Omega = \mathbb{R}^n$. Let $0 < \gamma < 1$. Then there exists a constant $c_3 = c_3(\Omega, \gamma) > 0$ such that for any $V \subset A \subset \Omega$ one has*

$$\int_{\Omega \setminus A} \int_V \frac{dx dy}{|x - y|^{n+1}} \geq c_3 |V|^{\frac{n-1}{n}} \quad (2)$$

provided that $|A| < \infty$,

$$|\Omega \setminus A| > |A| \quad (3)$$

and

$$|V| \geq \gamma |A|. \quad (4)$$

Remark 1. *The following example shows the importance of the restriction that $\Omega = \phi(B_1)$, where the diffeomorphism ϕ satisfies (1). The domain*

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < x_1^2\}$$

does not satisfy the conditions of Theorem 1. Let us take

$$A_k = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < \frac{2}{k}, 0 < x_2 < x_1^2\},$$

$$V_k = \{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < \frac{1}{k}, 0 < y_2 < y_1^2\}.$$

Note that $|\Omega \setminus A_k| > |A_k|$, $|V_k| = |A_k|/8$ for $k \geq 4$ and so (3) and (4) hold. One readily sees that

$$|x - y| \geq |x_1 - y_1| \geq \left| x_1 - \frac{1}{k} \right| \geq \frac{|x_1|}{2}$$

for $x = (x_1, x_2) \in \Omega \setminus A_k$, $y = (y_1, y_2) \in V_k$, where we have used $0 < y_1 < 1/k < 2/k < x_1$. Hence

$$\int_{\Omega \setminus A_k} \int_{V_k} \frac{dxdy}{|x - y|^3} \geq 8|V_k| \int_{\Omega \setminus A_k} \frac{dx}{|x_1|^3} = \frac{8}{3}|V_k| \left(1 - \frac{2}{k}\right).$$

Therefore

$$|V_k|^{\frac{-n+1}{n}} \int_{\Omega \setminus A_k} \int_{V_k} \frac{dxdy}{|x - y|^3} \rightarrow 0$$

as $k \rightarrow \infty$.

Proof of Lemma 1. Let us briefly outline the main ideas of the proof. The proof for the case $\Omega = \mathbb{R}^n$ is based on the fact that the expression

$$|V|^{\frac{-n+1}{n}} \int_{\mathbb{R}^n \setminus A} \int_V \frac{dxdy}{|x - y|^{n+1}}$$

is invariant under the replacement $(A, V) \rightarrow (tA, tV)$, where $t > 0$ is a parameter. In the case $\Omega = B_1$ we construct sets $\widehat{V} \subset \widehat{A}$, such that $A \subset \widehat{A}$, $V \subset \widehat{V}$ and $|\widehat{V}| \geq \text{const} \cdot |\widehat{A}| > 0$. Then we reduce this case to the previous one by applying Lemma 1 with $(\Omega, A, V) := (\mathbb{R}^n, \widehat{A}, \widehat{V})$. In the most general case $\Omega = \phi(B_1)$ it is enough to notice that both sides of (2) are compatible with the corresponding expressions for $\Omega = B_1$. Now we proceed to the proof.

Case 1. Let $\Omega = \mathbb{R}^n$.

Take a set of unit cubes $Q_m \subset \mathbb{R}^n$, $m \in \mathbb{N}$ such that $\mathbb{R}^n = \bigcup_{m=1}^{\infty} Q_m$.

Using that $|x - y| \leq \sqrt{n}$ for $x, y \in Q_m$ we obtain

$$\begin{aligned} \int_{A^c} \int_V \frac{dxdy}{|x - y|^{n+1}} &\geq \sum_{m=1}^{\infty} \int_{Q_m \cap A^c} \int_{Q_m \cap V} \frac{dxdy}{|x - y|^{n+1}} \\ &\geq \frac{1}{(\sqrt{n})^{n+1}} \sum_{m=1}^{\infty} |Q_m \cap A^c| |Q_m \cap V|, \end{aligned}$$

where $A^c = \mathbb{R}^n \setminus A$. If $|A| = 1/2$, then $|Q_m \cap A^c| \geq 1/2$ and using (4) we get

$$\int_{A^c} \int_V \frac{dxdy}{|x-y|^{n+1}} \geq \frac{1}{2(\sqrt{n})^{n+1}} \sum_{m=1}^{\infty} |Q_m \cap V| = \frac{|V|}{2(\sqrt{n})^{n+1}} \geq \frac{\gamma}{4(\sqrt{n})^{n+1}}. \quad (5)$$

In the general case we take $t = (2|A|)^{1/n}$ and $A_0 = t^{-1}A$, $V_0 = t^{-1}V$. Clearly $|A_0| = 1/2$ and $|V_0| \geq \gamma|A_0|$. Making the change of variables $x := tx$, $y := ty$ and applying (5) with A_0 and V_0 gives

$$\begin{aligned} \int_{A^c} \int_V \frac{dxdy}{|x-y|^{n+1}} &= t^{n-1} \int_{A_0^c} \int_{V_0} \frac{dxdy}{|x-y|^{n+1}} \geq t^{n-1} \frac{\gamma}{4(\sqrt{n})^{n+1}} \\ &\geq \frac{\gamma}{4(\sqrt{n})^{n+1}} |V|^{\frac{n-1}{n}}, \end{aligned}$$

where we used $t^{n-1} \geq (2|A|)^{\frac{n-1}{n}} \geq (2|V|)^{\frac{n-1}{n}} \geq |V|^{\frac{n-1}{n}}$.

Case 2. We proceed with the case $\Omega = B_1$. We denote by B_r the ball in \mathbb{R}^n with radius r and center at the origin. Let (A_k, V_k) be a sequence of sets minimizing $G = G(A, V)$

$$G := \frac{1}{|V|^{\frac{n-1}{n}}} \int_{\Omega \setminus A} \int_V \frac{dxdy}{|x-y|^{n+1}}.$$

among all (A, V) satisfying the conditions of Lemma 1. We are done if we show that

$$G(A_k, V_k) \geq \text{const.} > 0 \quad (6)$$

for all $k \in \mathbb{N}$.

Since $|x-y| \leq 2$ for $x, y \in B_1$, it follows that

$$G \geq 2^{-n-1} |\Omega \setminus A| |V|^{\frac{1}{n}} \geq 2^{-n-1} |A| |V|^{\frac{1}{n}}. \quad (7)$$

If $\inf_k |A_k| > 0$, then (6) is a straightforward consequence of (4) and (7). Therefore it suffices to consider the case $\lim_{k \rightarrow \infty} |A_k| = 0$. Without loss of generality we may assume that for any k

$$|A_k| < \frac{1}{2} |B_{\frac{1}{3}} \setminus B_{\frac{1}{4}}|. \quad (8)$$

Case 2a. Suppose that for all k

$$|V_k \cap B_{\frac{1}{2}}| > \frac{1}{2} |V_k|. \quad (9)$$

Since $|x - y| \leq 3/2 < 3$ for all $y \in B_{\frac{1}{2}}$, $x \in B_1 \setminus B_{\frac{3}{4}}$ and because of (8), it follows that

$$\int_{(B_1 \setminus B_{\frac{3}{4}}) \setminus A_k} \frac{dx}{|x - y|^{n+1}} \geq \frac{1}{3^{n+1}} |(B_1 \setminus B_{\frac{3}{4}}) \setminus A_k| \geq \frac{1}{3^{n+2}} |B_1 \setminus B_{\frac{3}{4}}| =: c_4 > 0.$$

Moreover, for any $y \in B_{\frac{1}{2}}$

$$\int_{\mathbb{R}^n \setminus B_1} \frac{dx}{|x - y|^{n+1}} \leq \int_{\mathbb{R}^n \setminus B_1} \frac{dx}{(|x| - \frac{1}{2})^{n+1}} =: c_5.$$

Thus for all $y \in B_{\frac{1}{2}}$ and $A_k \subset B_1$ satisfying (8) one has

$$\int_{(B_1 \setminus B_{\frac{3}{4}}) \setminus A_k} \frac{dx}{|x - y|^{n+1}} \geq c_6 \int_{(\mathbb{R}^n \setminus B_{\frac{3}{4}}) \setminus A_k} \frac{dx}{|x - y|^{n+1}}, \quad (10)$$

where $c_6 := c_4/(c_5 + c_4)$. Using

$$\int_{B_1 \setminus A_k} = \int_{B_{\frac{3}{4}} \setminus A_k} + \int_{(B_1 \setminus B_{\frac{3}{4}}) \setminus A_k}$$

and (10) we obtain

$$\begin{aligned} \int_{B_1 \setminus A_k} \int_{V_k \cap B_{\frac{1}{2}}} \frac{dx dy}{|x - y|^{n+1}} &\geq \int_{B_{\frac{3}{4}} \setminus A_k} \int_{V_k \cap B_{\frac{1}{2}}} \frac{dx dy}{|x - y|^{n+1}} + c_6 \int_{(\mathbb{R}^n \setminus B_{\frac{3}{4}}) \setminus A_k} \int_{V_k \cap B_{\frac{1}{2}}} \frac{dx dy}{|x - y|^{n+1}} \\ &\geq c_6 \int_{\mathbb{R}^n \setminus A_k} \int_{V_k \cap B_{\frac{1}{2}}} \frac{dx dy}{|x - y|^{n+1}}. \end{aligned}$$

Consequently we have

$$\int_{B_1 \setminus A_k} \int_{V_k} \frac{dx dy}{|x - y|^{n+1}} \geq \int_{B_1 \setminus A_k} \int_{V_k \cap B_{\frac{1}{2}}} \frac{dx dy}{|x - y|^{n+1}} \geq c_6 \int_{\mathbb{R}^n \setminus A_k} \int_{V_k \cap B_{\frac{1}{2}}} \frac{dx dy}{|x - y|^{n+1}}.$$

Using (9) we apply Lemma 1 with $(\Omega, A, V) := (\mathbb{R}^n, A_k, V_k \cap B_{\frac{1}{2}})$ to get

$$\int_{\mathbb{R}^n \setminus A_k} \int_{V_k \cap B_{\frac{1}{2}}} \frac{dxdy}{|x-y|^{n+1}} \geq c_3 |V_k \cap B_{\frac{1}{2}}|^{\frac{n-1}{n}} \geq c_3 2^{\frac{1-n}{n}} |V_k|^{\frac{n-1}{n}},$$

where $c_3 = c_3(\mathbb{R}^n, \gamma/2)$.

Case 2b. We suppose that for all k

$$|V_k \setminus B_{\frac{1}{2}}| \geq \frac{1}{2} |V_k|. \quad (11)$$

Denote

$$\omega(x) := \frac{x}{|x|^2}$$

and

$$V_k^0 := \omega(V_k \setminus B_{\frac{1}{2}}), \quad A_k^0 := \omega(A_k \setminus B_{\frac{1}{4}}), \quad (12)$$

$\widehat{V}_k = V_k^0 \cup V_k$ and $\widehat{A}_k = A_k^0 \cup A_k$. Clearly, $\widehat{V}_k \subset \widehat{A}_k \subset B_4$ and

$$(B_4 \setminus B_1) \setminus A_k^0 = \omega((B_1 \setminus B_{\frac{1}{4}}) \setminus A_k). \quad (13)$$

The elementary calculations show that

$$|\nabla \omega(x)| = \frac{1}{|x|^{2n}} \quad (14)$$

and

$$|\omega(x) - \omega(y)| \geq |x - y| \quad \text{if } |x|, |y| \leq 1 \quad (15)$$

$$|x - \omega(y)| \geq |x - y| \quad \text{if } |x|, |y| \leq 1. \quad (16)$$

Making the change of variables $x := \omega(x)$, $y := \omega(y)$ and applying (12), (13), (15) we obtain

$$\begin{aligned} \int_{(B_4 \setminus B_1) \setminus A_k^0} \int_{V_k^0} \frac{dxdy}{|x-y|^{n+1}} &= \int_{(B_1 \setminus B_{\frac{1}{4}}) \setminus A_k} \int_{V_k \setminus B_{\frac{1}{2}}} \frac{|\nabla \omega(x)| |\nabla \omega(y)| dxdy}{|\omega(x) - \omega(y)|^{n+1}} \\ &\leq 8^{2n} \int_{(B_1 \setminus B_{\frac{1}{4}}) \setminus A_k} \int_{V_k \setminus B_{\frac{1}{2}}} \frac{dxdy}{|x-y|^{n+1}} \\ &\leq 8^{2n} \int_{B_1 \setminus A_k} \int_{V_k} \frac{dxdy}{|x-y|^{n+1}}, \end{aligned} \quad (17)$$

where we have used, by (14),

$$|\nabla\omega(x)| \leq 4^{2n} \quad \text{for } x \in B_1 \setminus B_{\frac{1}{4}}, \quad |\nabla\omega(y)| \leq 2^{2n} \quad \text{for } y \in B_1 \setminus B_{\frac{1}{2}}.$$

Similarly, making the change of variables and using (12)-(14), (16) we have

$$\int_{(B_4 \setminus B_1) \setminus A_k^0} \int_{V_k} \frac{dxdy}{|x-y|^{n+1}} \leq 4^{2n} \int_{B_1 \setminus A_k} \int_{V_k} \frac{dxdy}{|x-y|^{n+1}} \quad (18)$$

and

$$\int_{B_1 \setminus A_k} \int_{V_k^0} \frac{dxdy}{|x-y|^{n+1}} \leq 2^{2n} \int_{B_1 \setminus A_k} \int_{V_k} \frac{dxdy}{|x-y|^{n+1}}. \quad (19)$$

Combining (17)-(19) we arrive at

$$\int_{B_4 \setminus \hat{A}_k} \int_{\hat{V}_k} \frac{dxdy}{|x-y|^{n+1}} \leq (1 + 2^{2n} + 4^{2n} + 8^{2n}) \int_{B_1 \setminus A_k} \int_{V_k} \frac{dxdy}{|x-y|^{n+1}}. \quad (20)$$

Since $|x-y| \leq 6$ for $x \in B_4 \setminus B_3$ and $y \in B_2$, it follows that

$$\int_{B_4 \setminus \hat{A}_k} \frac{dx}{|x-y|^{n+1}} \geq \int_{(B_4 \setminus B_3) \setminus A_k^0} \frac{dx}{|x-y|^{n+1}} \geq \frac{1}{6^{n+1}} |(B_4 \setminus B_3) \setminus A_k^0|$$

for all $y \in B_2$. In view of (8) and (14),

$$|(B_4 \setminus B_3) \setminus A_k^0| = |\omega((B_{\frac{1}{3}} \setminus B_{\frac{1}{4}}) \setminus A_k)| \geq |(B_{\frac{1}{3}} \setminus B_{\frac{1}{4}}) \setminus A_k| \geq \frac{1}{2} |B_{\frac{1}{3}} \setminus B_{\frac{1}{4}}|.$$

Thus

$$\int_{B_4 \setminus \hat{A}_k} \frac{dx}{|x-y|^{n+1}} \geq c_7 > 0,$$

where $c_7 = 6^{-n-2} |B_{\frac{1}{3}} \setminus B_{\frac{1}{4}}|$. Moreover, for any $y \in B_2$

$$\int_{\mathbb{R}^n \setminus B_4} \frac{dx}{|x-y|^{n+1}} \leq \int_{\mathbb{R}^n \setminus B_4} \frac{dx}{(|x|-2)^{n+1}} =: c_8.$$

Thus for all $y \in B_2$ one has

$$\int_{B_4 \setminus \hat{A}_k} \frac{dx}{|x-y|^{n+1}} \geq c_9 \int_{\mathbb{R}^n \setminus \hat{A}_k} \frac{dx}{|x-y|^{n+1}}, \quad (21)$$

where $c_9 = c_7/(c_8 + c_7)$. Combining (20), (21) and using $\widehat{V}_k \subset B_2$ we arrive at

$$\int_{B_1 \setminus A_k} \int_{V_k} \frac{dxdy}{|x-y|^{n+1}} \geq c_{10} \int_{\mathbb{R}^n \setminus \widehat{A}_k} \int_{\widehat{V}_k} \frac{dxdy}{|x-y|^{n+1}}, \quad (22)$$

where $c_{10} = c_9/(1 + 2^{2n} + 4^{2n} + 8^{2n})$. Observe that

$$|V_k^0| \geq |V_k \setminus B_{\frac{1}{2}}| \geq \frac{1}{2} |V_k|, \quad |A_k^0| \leq 4^{2n} |A_k \setminus B_{\frac{1}{4}}| \leq 4^{2n} |A_k|.$$

Consequently,

$$|V_k^0| \geq \frac{1}{2} |V_k| \geq \frac{\gamma}{2} |A_k| \geq \gamma 4^{-2n-1} |A_k^0|$$

and so

$$|\widehat{V}_k| \geq \gamma 4^{-2n} |\widehat{A}_k|.$$

Using this we apply Lemma 1 with $\Omega = \mathbb{R}^n$, $A = \widehat{A}_k$, $V = \widehat{V}_k$ to get

$$\int_{\mathbb{R}^n \setminus \widehat{A}_k} \int_{\widehat{V}_k} \frac{dxdy}{|x-y|^{n+1}} \geq c_3(\mathbb{R}^n, \gamma 4^{-2n}) |\widehat{V}_k|^{\frac{n-1}{n}} \geq c_3(\mathbb{R}^n, \gamma 4^{-2n}) |V_k|^{\frac{n-1}{n}}.$$

This and (22) prove Lemma 1 for Case 2b.

Case 3. Let $\Omega = \phi(B_1)$, where ϕ satisfies the conditions of Theorem 1.

Making the change of variables $x := \phi(x)$, $y := \phi(y)$ we get

$$\int_{\Omega \setminus A} \int_{\tilde{V}} \frac{dxdy}{|x-y|^{n+1}} = \int_{B_1 \setminus \tilde{A}} \int_{\tilde{V}} \frac{|\nabla \phi(x)| |\nabla \phi(y)| dxdy}{|\phi(x) - \phi(y)|^{n+1}},$$

where $\tilde{A} = \phi^{-1}(A)$, $\tilde{V} = \phi^{-1}(V)$. Because of (1) it follows that

$$|\nabla \phi(x)| \geq c_1^{-1}.$$

Hence

$$\int_{\Omega \setminus A} \int_{\tilde{V}} \frac{dxdy}{|x-y|^{n+1}} \geq c_1^{-(n+3)} \int_{B_1 \setminus \tilde{A}} \int_{\tilde{V}} \frac{dxdy}{|\phi(x) - \phi(y)|^{n+1}}.$$

Since $|\tilde{V}| \geq c_1^{-1} |V| \geq c_1^{-1} \gamma |A| \geq c_1^{-2} \gamma |\tilde{A}|$, an application of Lemma 1 with $(\Omega, A, V) := (B_1, \tilde{A}, \tilde{V})$ gives

$$\int_{\Omega \setminus A} \int_{\tilde{V}} \frac{dxdy}{|x-y|^{n+1}} \geq c_1^{-(n+3)} c_3 |\tilde{V}| \geq c_1^{-(n+4)} c_3 |V|,$$

where $c_3 = c_3(B_1, c_1^{-2} \gamma)$. □

Lemma 2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and suppose that $f : \Omega \rightarrow \mathbb{C}$ be a continuous function, which satisfies conditions of Theorem 1. Let*

$$I_x := \{y \in \Omega : |f(x)| > 2|f(y)|\}, \quad J_x := \{y \in \Omega : |f(y)| > 2|f(x)|\} \quad (23)$$

and

$$\psi(x) = \int_{I_x \cup J_x} \frac{dy}{|x - y|^{n+1}}. \quad (24)$$

Then

$$\int_{\Omega} |f(x)|^2 \psi(x) dx + \int_{\Omega} |f(x)|^2 dx \geq c_{11} \left(\int_{\Omega} |f(x)|^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} \quad (25)$$

for some $c_{11} = c_{11}(\Omega) > 0$.

Proof. Without loss of generality we may assume that $|f(x)| \leq 1$ for all $x \in \Omega$.

For any $m \in \mathbb{N}$ we put

$$D_m := \{x \in \Omega : f(x) \in [2^{-m}, 2^{-m+1}]\}$$

and $D_{\infty} = f^{-1}(0)$. Clearly, $\Omega = D_{\infty} \cup \bigcup_{m=1}^{\infty} D_m$. One has

$$\int_{\Omega} |f(x)|^{\frac{2n}{n-1}} dx \leq \sum_{m=1}^{\infty} \frac{|D_m|}{N^{m-1}}, \quad (26)$$

where $N = 2^{\frac{2n}{n-1}}$. Denote by $E \subset \mathbb{N}$ the set of indexes m such that

$$|D_m| \geq \frac{|D_{m-1}| + |D_{m+1}|}{64}. \quad (27)$$

Since

$$\begin{aligned} \frac{|D_{m-1}| + |D_{m+1}|}{64N^{m-1}} &\leq \frac{\max\{N, N^{-1}\}}{64} \left(\frac{|D_{m-1}|}{N^{m-2}} + \frac{|D_{m+1}|}{N^m} \right) \\ &\leq \frac{1}{4} \left(\frac{|D_{m-1}|}{N^{m-2}} + \frac{|D_{m+1}|}{N^m} \right), \end{aligned}$$

it follows that

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{|D_m|}{N^{m-1}} &= \sum_{m \in E} \frac{|D_m|}{N^{m-1}} + \sum_{m \in \mathbb{N} \setminus E} \frac{|D_m|}{N^{m-1}} \\
&\leq \sum_{m \in E} \frac{|D_m|}{N^{m-1}} + \sum_{m \in \mathbb{N} \setminus E} \frac{|D_{m-1}| + |D_{m+1}|}{64N^{m-1}} \\
&\leq \sum_{m \in E} \frac{|D_m|}{N^{m-1}} + \frac{1}{2} \sum_{m=1}^{\infty} \frac{|D_m|}{N^{m-1}},
\end{aligned}$$

where we have used that $64|D_m| < |D_{m-1}| + |D_{m+1}|$ for $m \in \mathbb{N} \setminus E$. Hence

$$\sum_{m=1}^{\infty} \frac{|D_m|}{N^{m-1}} \leq 2 \sum_{m \in E} \frac{|D_m|}{N^{m-1}}. \quad (28)$$

Consequently, from (26) and (28) we deduce that

$$\left(\int_{\Omega} |f(x)|^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq 2^{\frac{n-1}{n}} \sum_{m \in E} \frac{|D_m|^{\frac{n-1}{n}}}{4^{m-1}} \leq 8 \sum_{m \in E} \frac{|D_m|^{\frac{n-1}{n}}}{4^m}, \quad (29)$$

where we have used that $N^{\frac{n-1}{n}} = 4$ and the elementary inequality

$$\left(\sum_{k=1}^{\infty} a_k \right)^{\frac{n-1}{n}} \leq \sum_{k=1}^{\infty} a_k^{\frac{n-1}{n}} \quad (30)$$

for all $a_k \geq 0$.

We split the remaining part of the proof into two cases.

Case 1. Suppose that there exists $p \geq 2$ such that

$$|D_{p-1}| + |D_p| + |D_{p+1}| > \sum_{\substack{m=1 \\ m \neq p-1, p, p+1}}^{\infty} |D_m|. \quad (31)$$

From definition (23) we see that $\bigcup_{k=1}^{m-2} D_k \subset J_x$ and $D_{\infty} \cup \bigcup_{k=m+2}^{\infty} D_k \subset I_x$.

Using (24) we obtain

$$\begin{aligned}
\int_{\Omega} |f(x)|^2 \psi(x) dx &\geq \sum_{m=1}^{\infty} \int_{D_m} \int_{I_x \cup J_x} |f(x)|^2 \frac{dx dy}{|x-y|^{n+1}} \\
&\geq \sum_{\substack{m=1 \\ m \neq p-2, \dots, p+2}}^{\infty} \frac{1}{4^m} \int_{D_m} \int_{L_m} \frac{dx dy}{|x-y|^{n+1}}, \quad (32)
\end{aligned}$$

where

$$L_m := D_\infty \cup \bigcup_{\substack{k=1 \\ k \neq m-1, m, m+1}}^{\infty} D_k, \quad m \in \mathbb{N}. \quad (33)$$

From (31) we find

$$|L_m| \geq |D_{p-1}| + |D_p| + |D_{p+1}| \geq \frac{|\Omega|}{2} \quad (34)$$

for $m \neq p-2, p-1, p, p+1, p+2$. Hence (27) and (34) imply (4) and (3) respectively with $A = D_{m-1} \cup D_m \cup D_{m+1}$, $V = D_m$ and $\gamma = 1/65$. An application of Lemma 1 gives

$$\int_{D_m} \int_{L_m} \frac{dx dy}{|x-y|^{n+1}} \geq c_3 \frac{|D_m|^{\frac{n-1}{n}}}{4^m},$$

where $c_3 = c_3(\Omega, 1/65)$. Therefore

$$\int_{\Omega} |f(x)|^2 \psi(x) dx \geq c_3 \sum_{\substack{m=1 \\ m \neq p-2, \dots, p+2}}^{\infty} \frac{|D_m|^{\frac{n-1}{n}}}{4^m} \geq c_3 \sum_{\substack{m \in E \\ m \neq p-2, \dots, p+2}} \frac{|D_m|^{\frac{n-1}{n}}}{4^m}. \quad (35)$$

Using the second inequality in (34) we have

$$\begin{aligned} \int_{\Omega} |f(x)|^2 dx &\geq \sum_{m=p-2}^{p+2} \frac{|D_m|}{4^m} \geq \frac{\sum_{m=p-2}^{p+2} |D_m|}{4^{p+2}} \geq \left(\frac{|\Omega|}{2} \right)^{\frac{1}{n}} \frac{\left(\sum_{m=p-2}^{p+2} |D_m| \right)^{\frac{n-1}{n}}}{4^{p+2}} \\ &\geq \frac{1}{5} \left(\frac{|\Omega|}{2} \right)^{\frac{1}{n}} \frac{\left(\sum_{m=p-2}^{p+2} |D_m|^{\frac{n-1}{n}} \right)}{4^{p+2}} \\ &\geq c_{12} \sum_{m=p-2}^{p+2} \frac{|D_m|^{\frac{n-1}{n}}}{4^m}, \end{aligned} \quad (36)$$

where

$$c_{12} = \min \left\{ c_3, \frac{1}{4^4} \frac{1}{5} \left(\frac{|\Omega|}{2} \right)^{\frac{1}{n}} \right\}.$$

Piecing together (35) and (36) we have

$$\int_{\Omega} |f(x)|^2 \psi(x) dx + \int_{\Omega} |f(x)|^2 dx \geq c_{12} \sum_{m \in E} \frac{|D_m|^{\frac{n-1}{n}}}{4^m}.$$

Combining this and (29) we arrive at (25) with $c_{11} = c_{12}/8$.

Case 2. Let us assume that

$$|D_{m-1}| + |D_m| + |D_{m+1}| \leq \sum_{\substack{k=1 \\ k \neq m-1, m, m+1}} |D_k|$$

for all $m \geq 2$. Then for all m we have

$$|L_m| \geq \frac{|\Omega|}{2},$$

where L_m is defined by (33). As in Case 1 we apply Lemma 1 to get

$$\int_{\Omega} |f(x)|^2 \psi(x) dx \geq c_3 \sum_{m \in E} \frac{|D_m|^{\frac{n-1}{n}}}{4^m}. \quad (37)$$

Comparing (29) and (37) finishes the proof. \square

Proof of Theorem 1. As was shown in [BT] (consequence of Theorem 2 p.15) for any domain Ω satisfying conditions of Theorem 1 and for any continuous function $f : \Omega \rightarrow \mathbb{C}$ we have

$$\int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{n+1}} dx dy \geq c_{13} \int_{\Omega} |f(x)|^2 dx \quad (38)$$

for some constant $c_{13} = c_{13}(\Omega) > 0$. On the other hand

$$\int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{n+1}} dx dy \geq \int_{\Omega} \left(\int_{I_x \cup J_x} \frac{|f(x) - f(y)|^2}{|x - y|^{n+1}} dy \right) dx,$$

where I_x and J_x are given by (23). Since

$$|f(x) - f(y)| \geq \frac{|f(x)|}{2}$$

for all $y \in I_x \cup J_x$, it follows that

$$\begin{aligned} \int_{\Omega \times \Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{n+1}} dx dy &\geq \frac{1}{4} \int_{\Omega} |f(x)|^2 \left(\int_{I_x \cup J_x} \frac{dy}{|x - y|^{n+1}} \right) dx \\ &= \frac{1}{4} \int_{\Omega} |f(x)|^2 \psi(x) dx. \end{aligned} \quad (39)$$

Summing (38), (39) and applying Lemma 2 completes the proof. \square

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